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ON THE MINIMUM EFFORT REGULATION OF STATIONARY LINEAR SYSTEMS*

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by

I. Flügge-Lotz
Professor of Engineering Mechanics
and of Aeronautics and Astronautics
Stanford University
Stanford, California

8299000
4185252

and

H. D. Marbach (Hughes Aircraft Co.)
Hughes Aircraft Company
Space Systems Division
El Segundo, California

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SUMMARY

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The general problem considered is the restoring of a perturbed system to its equilibrium state with least expenditure of effort by the controlling elements. The measure of effort introduced encompasses cases in which the magnitude of effort expenditure does not depend explicitly on the state of the controlled system.

It is shown that minimum-effort control is in principle defined by a time-varying feedback process. Closed-form solutions are given for several examples involving a minimum-fuel criterion, and an example involving a quadratic measure of effort. In each case treated the minimum-effort control is obtained as a time-varying function of the instantaneous state of the system.

INTRODUCTION

Many present-day applications require that a system be capable of operating over extended periods with a limited supply of energy for control purposes. In such cases the efficient management of some exhaustible quantity such as fuel, electrical energy, propellant, etc., becomes a critical design problem.

The purpose here is to investigate control strategies which require minimum effort, where effort is understood to represent some specific consumable quantity of interest. Since closed-loop or feedback control techniques are generally preferable to open-loop techniques, the aim will be to derive the optimal strategy in the form of a feedback control law. A solution in this form may enhance the possibility that an optimal strategy can actually be implemented. In cases where such an implementation is not feasible, the optimal strategy may even so provide valuable insight into ways of synthesizing a suitable sub-optimal control law. In any case, a knowledge of the optimal control strategy will define a definite lower bound on the effort required to accomplish a given task, and can, therefore, provide a sound basis of comparison for other control strategies.

I. Formulation of the Problem

The instantaneous state of a system will be described by an n-dimensional vector $\underline{x}(t)$. A system of the type being considered is described by the linear, ordinary differential equation governing its response to controlling inputs,

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t)\,, \tag{1}$$

where $\underline{u}(t)$ is the r-dimensional vector control function whose components are subject to the amplitude constraints, $|u_i(t)| \le 1$, $i = 1, 2, \ldots, r$. A and B are constant nxn and nxr matrices respectively, and the column vectors of B will be denoted by $\underline{b}_i$, $i = 1, 2, \ldots, r$. The state variables, $x_i(t)$, $i = 1, 2, \ldots, n$, are chosen such that they are continuous in time, even at discontinuities of $\underline{u}(t)$. It is assumed that all state variables are controlled by the vector $\underline{u}(t)$.

The requirement that the control inputs $u_i(t)$ be limited in amplitude is a common one in practice, reflecting the limitations of practical hardware elements. These constraints can be briefly expressed by requiring that at any instant, $\underline{u}(t) \in \Omega$, where Ω is a closed region (hyper-cube) in r-dimensional space. An additional requirement will be that $\underline{u}(t)$ is a piecewise continuous function of time. Any piecewise continuous function $\underline{u}(t)$ taking its values in Ω will be termed an admissible control.

It is supposed that the system must be transferred from a known initial state, $\underline{x}(t_o) = \underline{x}_o$, to a definite final state, $\underline{x}(t_f) = \underline{x}_f$, where $\underline{x}_f$ is assumed to be an equilibrium state of the uncontrolled system (i.e., $\underline{x}_f$ satisfies $A\,\underline{x}_f = \underline{0}$). There is no loss of generality in designating the final state as, $\underline{x}_f = \underline{0}$, the origin of state space. Hence, the control $\underline{u}(t)$ is required to influence the system (1) in a way which satisfies the boundary conditions,

$$\underline{x}(t_o) = \underline{x}_o\,, \qquad \underline{x}(t_f) = \underline{0}\quad . \tag{2}$$

An important aspect of the problem formulation given here is that the total time allowed for the transition $\underline{x}_o \longrightarrow \underline{0}$ is treated as a fixed parameter. Denote the total length of the transition interval $[t_o, t_f]$ by T_o, where $T_o = t_f - t_o$. Then, this amounts to fixing the final instant as $t_f = t_o + T_o$. Of course, the value chosen for T_o cannot be less than the minimum time required for the transition $\underline{x}_o \longrightarrow \underline{0}$. Corresponding to a specific choice for T_o there exists a minimum-time isochrone (hereafter termed the T_o-isochrone) enclosing all states which can be restored to $\underline{0}$ by an admissible control in a time interval of length T_o. The initial state $\underline{x}_o$ must therefore lie inside the T_o-isochrone, or equivalently, T_o must be chosen large enough to include $\underline{x}_o$ within the isochrone.

Under the assumption that x_o lies inside the T_o-isochrone, there exist many admissible controls which accomplish the transition $\underline{x}_o \longrightarrow \underline{0}$ in the allowed time T_o. The problem is to find one which is best according to a specific criterion.

It is supposed that, to every transition of the system from $\underline{x}_o$ to $\underline{0}$, there is associated a number, denoted by E, which represents the total effort expended by the control in effecting this transition. For the present, the quantity E will be defined by an integral of the type,

$$E = \int_{t_o}^{t_f} \phi\ (\underline{u}(t))\ dt\ , \tag{3}$$

where $\phi\ (\underline{u})$ is a sufficiently smooth, non-negative, scalar-valued function of $\underline{u}$ with the property, $\phi(\underline{0}) = 0$. For a particular problem, E may represent, for example, the total energy or fuel consumed by the control elements of the system.

The problem considered here is the following: from among all control functions $\underline{u}(t)$ which cause the boundary conditions (2) to be satisfied, find one which yields the smallest value of E. Such a control function will be called an optimal control.

Let E^* denote the value imparted to the integral (3) by an optimal control. It is clear that this value will in general depend on the allowed time T_o as well as the initial state $\underline{x}_o$. That is $E^* = E^*(x_o, T_o)$. In fact, one can readily show that,

$$E^* (x_o, T_o + \epsilon) \leq E^* (x_o, T_o), \text{ for all } \epsilon > 0. \tag{4}$$

This means that, by increasing the time allowed for the transition $\underline{x}_o \longrightarrow \underline{0}$, the optimal value of E may decrease, but can never increase. Inequality (4) also explains why T_o must be regarded as a fixed parameter in this problem. With T_o left unspecified, (4) implies that a minimization of E would not be possible in general. On the other hand, fixing T_o does not restrict the generality of the results since the dependence of E^* on T_o can be determined once the fixed-time problem is solved.

II. Procedure for Deriving the Optimal Control Logic

The Pontryagin maximum principle ([1], [2], [8]) will be used to derive a necessary condition for optimality of a control $\underline{u}(t)$. Corresponding to the problem described above, the Hamiltonian function can be written as,*

$$H(\underline{x}(t), \underline{p}(t), \underline{u}(t)) = \underline{p}^t(t) \left[A\, \underline{x}(t) + B\, \underline{u}(t)\right] - \phi(\underline{u}(t)), \tag{5}$$

where $\underline{p}(t)$ is an n-dimensional vector function satisfying the adjoint differential equation,

$$\dot{\underline{p}}(t) = -A^t\, \underline{p}(t) \quad . \tag{6}$$

Since boundary conditions on $\underline{p}(t)$ are not specified, equation (6) does not define a unique vector function. Let $\underline{u}^*(t)$ be an admissible control function, and $\underline{x}^*(t)$ the resulting trajectory (solution of (1)) emanating from $\underline{x}_o$. According to the maximum principle, if $\underline{u}^*(t)$ is an optimal control, there exists a function $\underline{p}^*(t)$ satisfying (6) such that for every instant t of the control interval $\left[t_o, t_f\right]$, $H(\underline{x}^*(t), \underline{p}^*(t), \underline{u}^*(t))$ achieves the maximum value of the Hamiltonian (5) with respect to all admissible controls. For the present problem, this condition can be expressed as,

$$\left[\underline{p}^{*t}(t)\, B\, \underline{u}^*(t) - \phi(\underline{u}^*(t))\right] = \max_{\underline{u} \in \Omega} \left\{\underline{p}^t(t)\, B\, \underline{u}(t) - \phi(\underline{u}(t))\right\} \tag{7}$$

where (7) must hold at every instant of the interval $\left[t_o, t_f\right]$.

* The superscript t denotes the transpose of a vector or matrix.

Thus, if $\underline{u}^*(t)$ is an optimal control, the maximum principle guarantees that a function $\underline{p}(t)$ exists such that, at any instant $t \in [t_o, t_f]$, the value of $\underline{u}^*(t)$ can be determined as that value of $\underline{u}(t)$ which maximizes the expression,

$$P(\underline{p}(t), \underline{u}(t)) = \left[\underline{p}^t(t) B \underline{u}(t) - \phi(\underline{u}(t)) \right] . \tag{8}$$

The question of uniqueness arises here. That is, at a given instant there may be more than one value of $\underline{u}(t)$ which (corresponding to some vector $\underline{p}(t)$) achieves the maximum of $P(\underline{p}(t), \underline{u}(t))$. Hence, for this problem in general, an optimal control may not be unique. However, in specific cases where $\underline{u}^*(t)$ uniquely achieves the maximum of $P(\underline{p}(t), \underline{u}(t))$ for all t on $[t_o, t_f]$, $\underline{u}^*(t)$ will be the unique optimal control.*

It is convenient to define a particular function $\underline{p}(t)$ in terms of its value at the instant t_f, where the notation $\underline{p}_f = \underline{p}(t_f)$ will be used. Thus, any solution of the adjoint equation (6) can be expressed as,

$$\underline{p}(t) = e^{A^t T} \underline{p}_f , \tag{9}$$

where, $T = t_f - t$, is the reverse-time measured from t_f. The variable T can be thought of as being the time-to-go, or time remaining before the final instant is reached. With (9), the function P defined by (8) can be expressed as,

$$P(\underline{p}_f, T, \underline{u}) = \left[\underline{p}_f^t \, e^{A T} B \underline{u} - \phi(\underline{u}) \right] . \tag{10}$$

Any optimal control satisfies the maximum principle (7) with respect to some adjoint function $\underline{p}(t)$, or equivalently, with respect to some vector $\underline{p}_f$. Corresponding to a fixed $\underline{p}_f$, the control which satisfies the maximum principle is obtained from (10) as,

$$\underline{u}(\underline{p}_f, T) = \underset{\underline{u} \in \Omega}{\operatorname{argmax}} \; P(\underline{p}_f, T, \underline{u}) . \tag{11}$$

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* This follows from an argument similar to one employed in [1], sec. 18, to prove uniqueness of minimum-time controls for linear systems.

Hereafter it will be assumed that the matrix A and the function $\phi(\underline{u})$ are given such that (11) defines a unique vector $\underline{u}$ for any choice of the arguments $(\underline{p}_f, T)$. Under this assumption, all optimal controls will be unique. For problems excluded by this assumption, the following arguments can be amended to include only those optimal controls which are unique.

Since an optimal trajectory must intercept the origin $\underline{0}$ at the final instant t_f, the optimal trajectory corresponding to a control defined by (11) can be obtained by reverse-time integration of the system equations (1). The result of this integration is

$$\underline{x}(\underline{p}_f, T) = -\int_0^T e^{A(s-T)} B\, \underline{u}(\underline{p}_f, s)\, ds \quad . \tag{12}$$

For a particular choice of the arguments $(\underline{p}_f, T)$, (11) and (12) define two unique vectors, $\underline{u}(\underline{p}_f, T)$ and $\underline{x}(\underline{p}_f, T)$. Consider a fixed value of T, say T_1, $0 \leq T_1 \leq T_o$, representing a particular instant t_1 of the control interval, where $t_1 = t_f - T_1$. Then to every $\underline{p}_f$ there corresponds a $\underline{u}(\underline{p}_f, T_1)$ and an $\underline{x}(\underline{p}_f, T_1)$, via (11) and (12), representing possible instantaneous values of $\underline{u}(t_1)$ and $\underline{x}(t_1)$ for an optimal solution. Since all optimal controls are unique here, to every such state $\underline{x}(t_1)$ there will correspond only one vector $\underline{u}(t_1)$. If all $\underline{p}_f$ are taken into account, then all possible optimal combinations of $\underline{x}(t_1)$ and $\underline{u}(t_1)$ will be established by (11) and (12). Under the assumption that all such combinations can be tabulated, this means that, at the instant t_1, the optimal instantaneous value of $\underline{u}(t_1)$ is dependent only on the instantaneous state $\underline{x}(t_1)$. Moreover, at an arbitrary instant t of the control interval, the optimal choice of $\underline{u}(t)$ is dependent only on $\underline{x}(t)$ and T, the time-to-go at that instant.

Define an (n + r + 1)-dimensional vector-valued function $\underline{z}(\underline{p}_f, T)$ as,

$$\underline{z}(\underline{p}_f, T) = \underline{x}(\underline{p}_f, T) \times \underline{u}(\underline{p}_f, T) \times T \tag{13}$$

where x denotes the cartesian product. Then (13), with (11) and (12), defines a mapping from (n + 1)-space (occupied by $\underline{p}_f \times T$) to an (n + 1)-dimensional surface in (n + r + 1)-space. Figure (1) gives a conceptual illustration of a cross-section of this surface corresponding to a particular instant t_1. The intersection of the surface with the plane $T = T_1$, where $T_1 = t_f - t_1$, is denoted by $Z(T_1)$. As indicated in Figure (1), $Z(T_1)$ yields the optimal control logic at the instant t_1. That is, corresponding to an instantaneous state $\underline{x}(t_1)$, the instantaneous optimal control $\underline{u}(t_1)$ is defined by the point $\underline{x}(t_1) \times \underline{u}(t_1) \times T_1$ lying on $Z(T_1)$

This means that, for problems of the type being considered, an optimal control is in principle describable in the form,

$$\underline{u}(t) = \underline{v}(\underline{x}(t), T) \quad , \tag{14}$$

which implies a time-variable, or programmed, feedback control process such as that illustrated in Figure (2).

Determining the optimal control logic for a particular problem may be a difficult task, and it may not be possible to obtain $\underline{v}(\underline{x}(t), T)$ in closed form. The degree of difficulty is evidently dependent on the order of the system, the characteristic roots of A, the number of control inputs, and the performance criterion, which means the form of $\phi(\underline{u})$. Once this logic is determined, however, it will encompass all minimum-effort transitions in the system.

Several examples will now be considered to illustrate how the optimal control logic (14) can be derived in specific cases.

III. The Minimum Fuel Problem

The so-called minimum fuel problem is characterized by the performance integral,

$$E = \int_{t_o}^{t_f} \phi(\underline{u})\, dt = \int_{t_o}^{t_f} \left[\sum_{i=1}^{r} c_i \left| u_i(t) \right| \right] dt \qquad (15)$$

where, $c_i > 0$, $i = 1, 2, \ldots, r$. This problem arises, for example, when rockets or reaction-jets are used as controlling elements. In such cases the quantity E defined by (15) represents fuel or propellant consumption.

An optimal control is given in terms of some fixed vector $\underline{p}_f$ and the time-to-go variable, T, by relation (11). Since, in this case, $\phi(\underline{u})$ is a sum of terms, each of which depends on only one control variable, $P(\underline{p}_f, T, \underline{u})$ is maximized independently for each u_i, $i = 1, 2, \ldots, r$. There are two classes of problems to be considered here.

Class I: The characteristic roots of the system matrix A are all nonzero. Relation (11) yields, for $i = 1, 2, \ldots, r$,

$$u_i(\underline{p}_f, T) = \begin{cases} \operatorname{sgn}\left[\underline{p}_f^t\, e^{AT}\, \underline{b}_i\right], & \text{for} \left|\underline{p}_f^t\, e^{AT}\, \underline{b}_i\right| \geq c_i \\ 0, & \text{for} \left|\underline{p}_f^t\, e^{AT}\, \underline{b}_i\right| < c_i \end{cases} \qquad (16)$$

Class II: At least one characteristic root of A is zero. For this case one or more vectors $\underline{p}_f$ can be found such that for at least one value of the index i, $\underline{p}_f^t\, e^{AT}\, \underline{b}_i \equiv c_i$. This assertion depends on the system (1) being controllable, and is readily proved by straightforward means. When A has at least one zero characteristic root, then for some value of i the vector $e^{AT}\, \underline{b}_i$ will contain one element which is constant. An obvious choice for $\underline{p}_f$ then yields $\underline{p}_f^t\, e^{AT}\, \underline{b}_i \equiv c_i$.

Application of (11) in such cases yields,

$$\begin{aligned} &\text{(a)} \qquad u_i(\underline{p}_f, T) = 0, \\ \text{or}\quad &\text{(b)} \qquad \operatorname{sgn} u_i(\underline{p}_f, T) = \operatorname{sgn}\left[\underline{p}_f^t\, e^{AT}\, \underline{b}_i\right] \end{aligned} \tag{17}$$

That is, for any T, $u_i(\underline{p}_f, T)$ is allowed to satisfy either (a) or (b) of (17). For these values of $\underline{p}_f$, therefore, the maximum principle does not prescribe a unique control function.

For problems of Class I, (16) defines a unique function $\underline{u}(\underline{p}_f, T)$ for any choice of $\underline{p}_f$, and hence, all optimal controls are unique. From previous discussions this means that any optimal control is in principle obtainable as a time-variable function of the instantaneous state. Moreover, for this problem each component of an optimal control can be expressed as,

$$u_i(t) = v_i(\underline{x}(t), T), \quad i = 1, 2, \ldots, r. \tag{18}$$

From (16), each of these components can assume only the values +1, -1, or 0, and must therefore be piecewise constant and, in general, discontinuous with time. The task of determining the control logic $\underline{v}(\underline{x}(t), T)$ therefore reduces to finding, for each control variable $u_i(t)$, $i = 1, 2, \ldots, r$, those time-varying regions of state space corresponding to the three possible instantaneous values of $u_i(t)$. Two specific problems of Class I are discussed in the examples to follow

Problems of Class II require a slightly modified treatment. As indicated by (17), for some $\underline{p}_f$ the maximum principle gives only an ambiguous specification of $\underline{u}(\underline{p}_f, T)$ by (11). Non-unique optimal controls of the type which arise in such cases have been termed singular controls in the literature. ([2])

Despite the occurrence of singular controls for problems of Class II, one can nevertheless derive a control logic which always yields an optimal trajectory. By proceeding in the same manner as for problems of Class I, the logic, $\underline{v}(\underline{x}(t), T)$, yielding all unique solutions can be derived. And, in those time-varying regions of state space where the optimal value of a $u_i(t)$ is not unique, one can choose $u_i(t) = 0$. With this strategy, a trajectory starting in a region of non-unique control must at some instant enter a region of unique control.

The simple $1/s^2$ plant provides an interesting example of a Class II problem having application in the attitude control of satellites. A solution of this problem in the form of (14) was first presented by B. Friedland and H. Ladd in [3]. Other treatments can be found in [4], [5], [6], and [7].

<u>Example 1</u>

Let the system equations be given as,

$$\overset{\circ}{\underline{x}}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \underline{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \quad , \tag{19}$$

where $\underline{x}(t) = (x_1(t), x_2(t))^t$ describes the instantaneous state of the system, and $u(t)$ is the scalar control function. The characteristic roots of the system are obtained from the equation,

$$\det \left[\lambda I - A \right] = 0 \quad , \tag{20}$$

which yields for this case, $\lambda = \pm i$. Since there are no zero roots, this problem belongs to Class I.

Dynamical behavior of the system can be depicted by motion of $\underline{x}(t)$ in the phase plane, as illustrated in Figure (3). In forward time, $\underline{x}(t)$ follows a trajectory which proceeds in a clockwise direction about the instantaneous center-point, $\underline{x}_c = (u(t), 0)^t$. On intervals where $u(t)$ is constant, the elapsed time between two points on the trajectory arc is equal to the angle subtended by radii drawn to those points, as indicated in Figure (3). A point on the trajectory corresponding to an instant at which a piecewise-constant u(t) changes value is called a switching point.

The performance integral (15) becomes for this example,

$$E = \int_{t_o}^{t_f} \left| u(t) \right| \, dt \quad . \tag{21}$$

From (16), any optimal control is given in terms of T and some vector $\underline{p}_f$ by the switching rule,

$$u(\underline{p}_f, T) = \begin{cases} \operatorname{sgn}\left[\underline{p}_f^t \, e^{AT} \, \underline{b}\right], & \text{for } \left|\underline{p}_f^t \, e^{AT} \, \underline{b}\right| \geq 1 \\ 0 & \text{for } \left|\underline{p}_f^t \, e^{AT} \, \underline{b}\right| < 1 \end{cases} \tag{22}$$

The aim here is to derive the control law, $u(t) = v(\underline{x}(t), T)$. Consider an arbitrary instant t_1 of the control interval, and denote the time-to-go at that instant by T_1. Let P_f denote the plane in which the vector $\underline{p}_f = (p_{f1}, p_{f2})^t$ takes its value. Then, corresponding to the instant t_1, P_f can be partitioned into three mutually exclusive regions, $R_+(T_1)$, $R_-(T_1)$, and $R_o(T_1)$, which are defined as,

$$\begin{aligned} R_+(T_1) \; &: \quad \text{all } \underline{p}_f \text{ such that, } \underline{p}_f^t \, e^{AT_1} \underline{b} \geq 1 \\ R_-(T_1) \; &: \quad \text{all } \underline{p}_f \text{ such that, } \underline{p}_f^t \, e^{AT_1} \underline{b} \leq -1 \\ R_o(T_1) \; &: \quad \text{all } \underline{p}_f \text{ such that, } -1 < \underline{p}_f^t \, e^{AT_1} \, \underline{b} < 1 \end{aligned} \tag{23}$$

The division of P_f into these three regions is illustrated in Figure (4), where the boundaries of $R_+(T_1)$ and $R_-(T_1)$ have been denoted by $S_+(T_1)$ and $S_-(T_1)$, respectively. With these definitions, and using (22), $u(\underline{p}_f, T_1)$ is obtained for any $\underline{p}_f$ by the following rule.

$$u(\underline{p}_f, T_1) = 1, \text{ if } \underline{p}_f \in R_+(T_1),$$
$$u(\underline{p}_f, T_1) = -1, \text{ if } \underline{p}_f \in R_-(T_1), \qquad (24)$$
$$u(\underline{p}_f, T_1) = 0, \text{ if } \underline{p}_f \in R_o(T_1).$$

(24) is interpreted as follows: The $\underline{p}_f$ lying in $R_+(T_1)$ correspond via (22) to all optimal controls which assume the value +1 at the instant t_1. Similarly, the $\underline{p}_f$ in $R_-(T_1)$ and $R_o(T_1)$ correspond to all optimal controls which assume the values -1 and 0, respectively, at t_1. Thus, in order to establish an instantaneous control law, $u(t_1) = v(\underline{x}(t_1), T_1)$, it remains to establish a correspondence between points in Γ_f and points in the state space (phase plane), which is denoted here by X. The desired mapping from P_f to X is provided by (12), which becomes for this case,

$$\underline{x}(\underline{p}_f, T_1) = -\int_0^{T_1} \begin{pmatrix} \sin(s - T_1) \\ \cos(s - T_1) \end{pmatrix} u(\underline{p}_f, s)\, ds. \qquad (25)$$

Under the mapping defined by (25), let $X_+(T_1)$, $X_-(T_1)$ and $X_o(T_1)$ be the images of $R_+(T_1)$, $R_-(T_1)$, and $R_o(T_1)$, respectively. Also let $L_+(T_1)$ and $L_-(T_1)$ be the images of $S_+(T_1)$ and $S_-(T_1)$, respectively. Then, before proceeding with the details of this mapping, the results obtained so far can be summarized as follows:

> At an arbitrary instant t_1 of the control interval, where, $t_1 = t_f - T_1$, the instantaneous value of an optimal control is given in terms of the instantaneous state, $\underline{x}(t_1)$, by the rule,

$$u(t_1) = \begin{cases} 1, & \text{if } \underline{x}(t_1) \in X_+(T_1) \\ 0, & \text{if } \underline{x}(t_1) \in X_o(T_1) \\ -1, & \text{if } \underline{x}(t_1) \in X_-(T_1) \end{cases} \qquad (26)$$

For the present problem it is sufficient to establish only the images of $S_+(T_1)$ and $S_-(T_1)$. It follows from (22) that $L_+(T_1)$ is the locus of all possible states at which an optimal control can switch between the values 0 and +1 at the instant t_1. Similarly, $L_-(T_1)$ gives all states at which a switching between the values 0 and -1 can occur. It also follows from (22) and (25) that $L_-(T_1)$ is the reflection of $L_+(T_1)$ about the origin of X. Thus, only $L_+(T_1)$ need actually be constructed here.

Let T_1 be decomposed as,

$$T_1 = k\pi + \xi \quad , \tag{27}$$

where, k is a positive integer or zero, and ξ is a number in the range, $0 \leqq \xi < \pi$. Corresponding to any point $\underline{p}_f$ lying on $S_+(T_1)$, the switching rule (22) yields a control function $u(\underline{p}_f, T)$ which can take either of the forms (a) or (b) shown in Figure (5). Note that the intervals on which a particular $u(\underline{p}_f, T)$ is nonzero are all of equal length Δ, $0 \leqq \Delta \leqq \pi$, where Δ depends only on $\| \underline{p}_f \|$. For any such $\underline{p}_f$ a state vector $\underline{x}(\underline{p}_f, T_1)$ is established from (25) by means of an elementary calculation. With Δ as a parameter, and taking all $\underline{p}_f$ on $S_+(T_1)$ into account, the results of these calculations can be tabulated as follows:

(a) $\quad u(t_1 + 0) = 1$

$$\underline{x}(t_1) = \begin{cases} (k+1)\begin{pmatrix} (1 - \cos \Delta) \\ - \sin \Delta \end{pmatrix}, & \text{for } 0 \leqq \Delta \leqq \xi \\[2ex] \begin{pmatrix} (k + 1 - \cos \xi - k \cos \Delta) \\ (-\sin \xi - k \sin \Delta) \end{pmatrix}, & \text{for } \xi \leqq \Delta \leqq \pi \end{cases}$$

(b) $\quad u(t_1 + 0) = 0$ (28)

$$\underline{x}(t_1) = \begin{cases} -k\begin{pmatrix}(1 - \cos\Delta)\\ \sin\Delta\end{pmatrix}, & \text{for } 0 \leqq \Delta \leqq \pi - \xi \\ \begin{pmatrix}(-k + \cos\xi + (k+1)\cos\Delta)\\ (\sin\xi - (k+1)\sin\Delta)\end{pmatrix}, & \text{for } \pi - \xi \leqq \Delta \leqq \pi \end{cases}$$

Given a specific value for T_1, and therefore definite values for k and ξ, (28) defines $L_+(T_1)$ as a locus of points in the phase plane, with Δ being a parameter along the locus.

For the sake of definiteness, let $T_1 = 3\pi/2$. Then $k = 1$ and $\xi = \pi/2$. With these values, (28) describes $L_+(T_1)$ as illustrated in Figure (6). Also shown are, $L_-(T_1)$, which is obtained by symmetry, the T_1-isochrone, and, for comparison purposes, the well-known Bushaw curve, which gives the minimum-time switching locus for this problem.* The union of $L_+(T_1)$ and $L_-(T_1)$ forms two closed curves which become the optimal switching-locus for the instant t_1. That is, an optimal control changes value at t_1 if and only if $\underline{x}(t_1)$ lies on this locus.

The region enclosed by the switching locus is evidently $X_0(T_1)$. Also, $X_+(T_1)$ and $X_-(T_1)$ are the regions bounded by the T_1-isochrone and $L_+(T_1)$ and $L_-(T_1)$, respectively. Figure (6) is therefore a graphical presentation of the instantaneous control law, $u(t_1) = v(\underline{x}(t_1), T_1)$, for $T_1 = 3\pi/2$. For any state $\underline{x}(t_1)$ lying inside the T_1-isochrone, the unique optimal value of $u(t_1)$ is prescribed according to the rule (26). Any state $\underline{x}(t_1)$ lying outside the T_1-isochrone cannot be restored to the origin at the final instant t_f, and hence there is no solution in such cases.

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* See [1] for a discussion of the minimum-time solution and a derivation of the Bushaw curve. Details for the construction of minimum-time isochrones are given in [5] and [8].

Similarly, for any instant the optimal value of u(t) is determined by the location of $\underline{x}(t)$ relative to the time-varying regions, $X_+(T)$, $X_-(T)$, and $X_o(T)$. Since an optimal control is piecewise constant, it is sufficient to determine only the instants at which the system trajectory crosses the time-varying switching curve. It is deduced from (28) that, with $T = k\pi + \xi$, the optimal switching-curve is constructed as shown in Figure (7). Since the curve is symmetrical about $\underline{0}$, only the portion in the left half-plane is shown.

As an optimal trajectory proceeds toward its eventual interception with $\underline{0}$ at the final instant, the optimal switching-curve continually contracts and finally, at t_f, it shrinks to the single point $\underline{0}$. Figure (8) illustrates how the switching curve propogates with T.

Example 2

In example 1 the minimum-fuel control logic was derived for a single input to a second-order plant. If two control inputs are involved, the same general procedure, with somewhat more lengthy calculations, leads to the minimum-fuel logic for each control variable. The object here will be merely to exhibit the solution of a two-input problem. A more detailed treatment can be found in [5].

For this case the system is described by,

$$\dot{\underline{x}}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \underline{x}(t) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underline{u}(t) \quad , \tag{29}$$

where $\underline{u}(t) = (u_1(t), u_2(t))^t$ is the two-dimensional, vector control function. The performance integral (15) will be taken as,

$$E = \int_{t_o}^{t_f} \left[|u_1(t)| + |u_2(t)| \right] dt \quad . \tag{30}$$

This is again a Class I problem. Any optimal control is therefore describable as,

$$\underline{u}(t) = \begin{pmatrix} v_1(\,\underline{x}(t),\ T\,) \\ \\ v_2(\,\underline{x}(t),\ T\,) \end{pmatrix} \quad . \tag{31}$$

From (16) the optimal control variables are always piecewise constant. Optimal phase-plane trajectories are therefore composed of circular arcs which move about the instantaneous center-point, $\underline{x}_c = (\,-\,u_1(t),\ u_2(t)\,)^t$.

By proceeding in the manner outlined in example 1, each control variable can be represented by a function of the instantaneous state and time-to-go. Figure (9) illustrates the optimal logic for $u_1(t)$ for the particular instant $T = 5\pi/4$. Corresponding to any state $\underline{x}(t)$ which lies inside or on the minimum-time isochrone, there is a unique optimal value of $u_1(t)$. Also shown in Figure (9) is the minimum-time switching locus ([1]). This curve has no bearing on the minimum-fuel control logic and is shown here only for comparison. Details for the construction of minimum-time isochrones (where the effects of both control variables must be included) are given in [5].

As time-to-go decreases toward zero, the various regions indicated in Figure(9) contract. The time-varying boundary enclosing states for which $u_1 = 0$ is the minimum-fuel switching curve for $u_1(t)$. A similar result is obtained for $u_2(t)$. The composite optimal control logic for both $u_1(t)$ and $u_2(t)$ is illustrated in Figure (10) for the instant, $T = 3\pi/2$. There are nine regions within the isochrone, each of which corresponds to a particular value of $\underline{u} = (u_1,\ u_2)^t$. As T decreases, these regions contract, and at any instant prescribe the optimal control vector $\underline{u}(t)$ as a function of the instantaneous state $\underline{x}(t)$.

IV. A Quadratic Effort Criterion

Consider next a class of problems for which control effort is defined by an integral of the type,

$$E = \frac{1}{2} \int_{t_o}^{t_f} \left[\sum_{i=1}^{r} c_i \, u_i(t)^2 \right] dt \quad , \tag{32}$$

where $c_i > 0$, $i = 1, 2, \ldots, r$.

For such problems it is found from (11) that the components of an optimal control are given, for $i = 1, 2, \ldots, r$, by,

$$u_i(\underline{p}_f, T) = \begin{cases} c_i^{-1} \left[\underline{p}_f^t \, e^{AT} \, \underline{b}_i \right] , & \text{for } \left| \underline{p}_f^t \, e^{AT} \, \underline{b}_i \right| < c_i \\ \operatorname{sgn} \left[\underline{p}_f^t \, e^{AT} \, \underline{b}_i \right] , & \text{for } \left| \underline{p}_f^t \, e^{AT} \, \underline{b}_i \right| \geq c_i \end{cases} \tag{33}$$

Since (33) defines a unique vector $\underline{u}$ for every choice of the arguments $(\underline{p}_f, T)$, any optimal control is unique for this problem. Hence, any optimal control can in principle be obtained via a programmed-feedback control law in the form of (18).

Note that, unlike the minimum-fuel case, the optimal control variables with criterion (32) are allowed to assume nonzero values which are less in magnitude than unity. Since (33) does not allow $u_i \equiv 0$ on any interval, periods of "coasting' cannot occur for this problem.

Example 3

Consider a system described by the equation,

$$\dot{\underline{x}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \underline{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \; , \tag{34}$$

with a quadratic effort index defined by,

$$E = \frac{1}{2} \int_{t_o}^{t_f} \left[u(t)\right]^2 \, dt \quad . \tag{35}$$

For this case (33) yields,

$$u(\underline{p}_f, T) = \begin{cases} (p_{f2} + T\, p_{f1}) & , \text{ for } \left| p_{f2} + T\, p_{f1} \right| < 1 \\ \operatorname{sgn}(p_{f2} + T\, p_{f1}) , & \text{ for } \left| p_{f2} + T\, p_{f1} \right| \geq 1 \end{cases} \tag{36}$$

Let T_1 be the time-to-go at some instant t_1, $t_o \leq t_1 < t_f$. If all possible choices of $\underline{p}_f = (p_{f1}, p_{f2})^t$ are considered, (36) defines all functions of T which are of the three types illustrated in Figure (11). Functions of type I are saturated at t_1, and become unsaturated at a subsequent instant of the control interval. They may or may not saturate again at the opposite limit. Type III functions are unsaturated on the entire control interval. Type II functions are unsaturated at t_1, but become saturated before the final instant is reached.

Reverse-time integration of the system equations is accomplished by (12), which becomes for this problem,

$$\underline{x}(\underline{p}_f, T) = - \int_0^T \begin{pmatrix} (s - T) \\ 1 \end{pmatrix} u(\underline{p}_f, s) \, ds \quad , \tag{37}$$

where T denotes an arbitrary value of time-to-go. By means of (37), controls of type I establish all states $\underline{x}(t)$ lying in the regions labeled I in Figure (12). These regions vary with T and yield all states corresponding to optimal controls which are instantaneously saturated. Similarly, controls of types II and III establish all states lying in regions II and III, respectively, in Figure (12).

The outer boundary in Figure (12) is the T-isochrone. As T decreases, this boundary contracts, and at any instant defines the region of all states which can be restored to $\underline{0}$ at the final instant by an admissible control function. For any state lying within this contracting isochrone there is a unique optimal value for u(t). It is not difficult to show that, with regions I, II, and III defined as in Figure (12), the optimal control logic for this problem is given by the following relations.

Region I: $u(t) = \pm 1$

Region II:
$$\left\{\begin{array}{ll} 0 \leqq x_2 \leqq T, & u(t) = \left[\dfrac{2}{3}\,\dfrac{(T - x_2)^2}{(x_1 + T^2/2)} - 1\right] \\[2ex] -T \leqq x_2 \leqq 0, & u(t) = \left[\dfrac{2}{3}\,\dfrac{(T + x_2)^2}{(x_1 - T^2/2)} + 1\right] \end{array}\right. \qquad (38)$$

Region III: $u(t) = -\dfrac{6}{T^2}\,x_1 - \dfrac{4}{T}\,x_2$

It is interesting to consider how a trajectory in Region III approaches the final point $(x_1, x_2)^t = (0, 0)^t$. With $u(\underline{p}_f, T)$ given by (36) ($|u| < 1$ in Region III), the integration (37) gives,

$$\left.\begin{aligned} x_1 &= p_{f2}\,\frac{T^2}{2} + p_{f1}\,\frac{T^3}{6} \\ x_2 &= -p_{f2}T - p_{f1}\,\frac{T^2}{2} \end{aligned}\right\} \qquad (39)$$

As an example suppose p_{f1} and p_{f2} are different from zero. For T very small, and providing that $\left(\dfrac{p_{f1}}{p_{f2}}\right) T \ll 1$, (39) yields,

$$x_1 \cong p_{f2}\,\frac{T^2}{2}, \qquad x_2 \cong -p_{f2}T,$$

which means that in this case the final portion of the trajectory approaches the parabola,

$$x_1 = \frac{x_2^2}{2p_{f2}} \quad .$$

The case $p_{f2} = 0$ is also interesting. For this case one obtains from (39),

$$|x_1| = \frac{1}{3}\sqrt{\frac{2}{|p_{f1}|}}\,|x_2|^{3/2} \quad .$$

CONCLUSION

In the regulation of stationary linear systems, the control strategy which achieves the minimum expenditure of control effort has been identified as a time-varying feedback process. The optimal control inputs were found to depend only on the instantaneous state of the controlled system, and the instantaneous time-to-go. Derivation of the control logic in a specific case may be a difficult task, depending on the complexity of the system, and a closed-form solution is not guaranteed. The general procedure for deriving the control logic has been outlined, and was used to obtain closed-form solutions for several examples.

The so-called minimum-fuel problem was discussed and the optimal control law derived in the specific case of a second-order plant with one control input. The solution for the problem with two inputs was exhibited. In both cases the minimum-fuel control logic is described by time-varying switching curves in the phase plane. Problems involving a quadratic measure of effort were briefly discussed and the optimal control law was obtained for a specific second-order example.

Strictly speaking, since the optimal feedback processes described here require an external time reference in order to determine time-to-go, they do not yield true closed-loop control. However, for a single transition of the system, the optimal strategy gives strictly a closed-loop control which is incidentally capable of handling small unexpected disturbances.

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FIGURE CAPTIONS

Fig. 1 - Determination of the instantaneous optimal control from the time-to-go and instantaneous state.

Fig. 2 - Schematic representation of optimal, programmed-feedback control process.

Fig. 3 - Phase-Plane representation of system motion. System $\frac{1}{s^2 + 1}$.

Fig. 4 - Partitioning of the plane P_f into three mutually exclusive regions.

Fig. 5 - Possible forms of control functions corresponding to points $\underline{p}_f$ lying on $S_+(T_1)$.

Fig. 6 - The optimal control logic for an instant when time-to-go is, $T = 3\pi/2$.

Fig. 7 - Construction of the optimal switching-curve for arbitrary time-to-go, where $T = k\pi + \xi$.

Fig. 8 - Propagation of the optimal switching-curve with time-to-go.

Fig. 9 - Instantaneous logic for determining $u_1(t)$ in the two-input case; $T = 5\pi/4$.

Fig. 10 - The composite optimal control logic for $\underline{u} = (u_1, u_2)$; $T = 3\pi/2$.

Fig. 11 - Possible forms of an optimal control function for the quadratic effort example. System $1/s^2$.

Fig. 12 - The time-varying optimal control law, $u = v(x, T)$, for the quadratic effort example. See Equation (38).

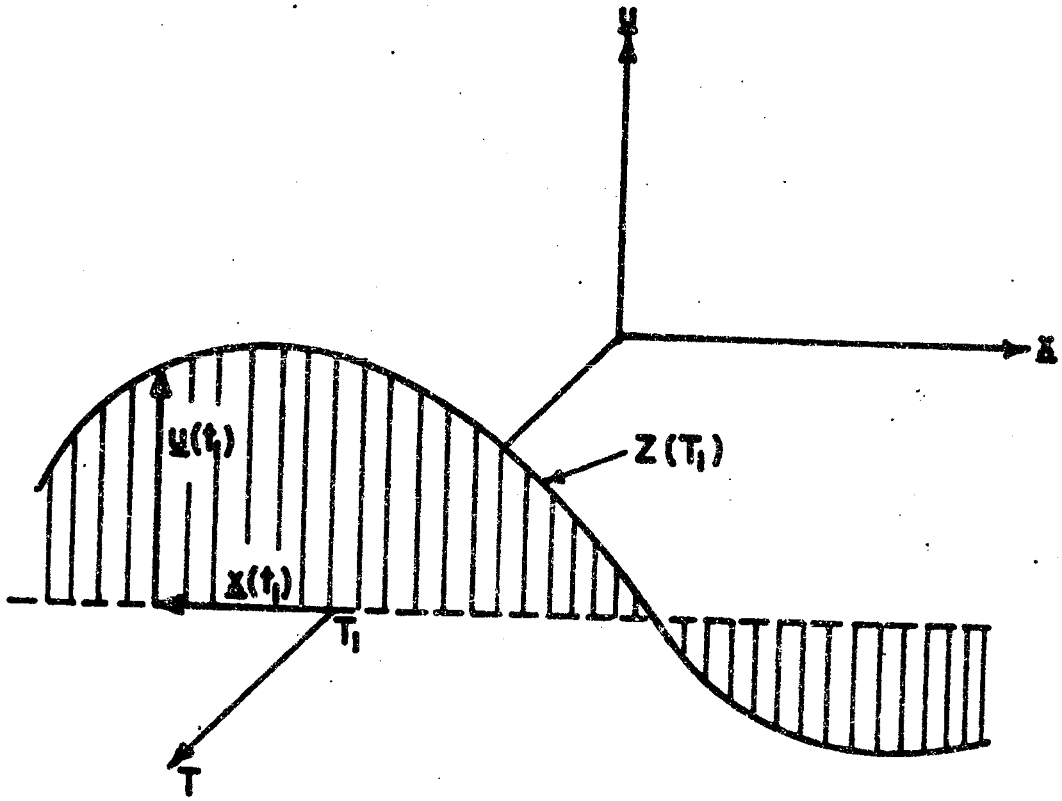


Fig 1 - Flügge-Lotz & MARBACH

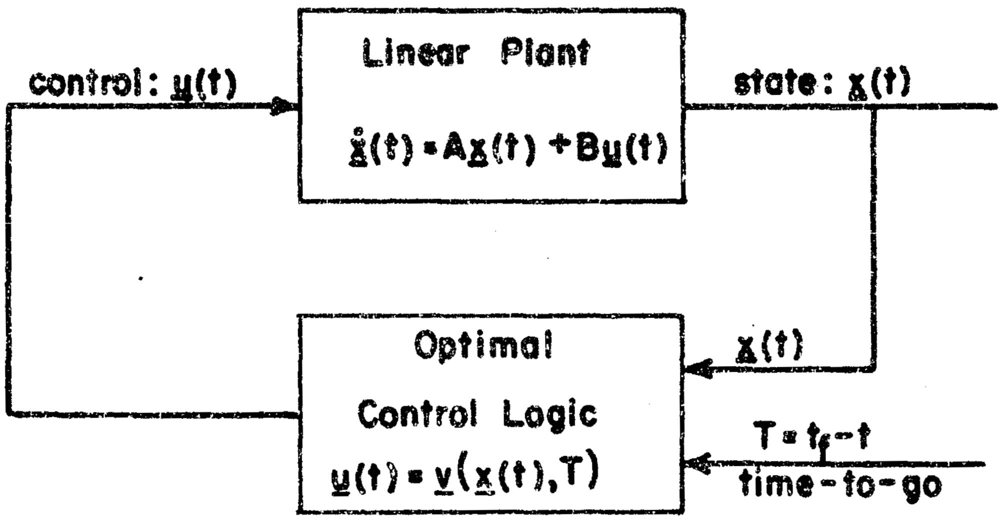
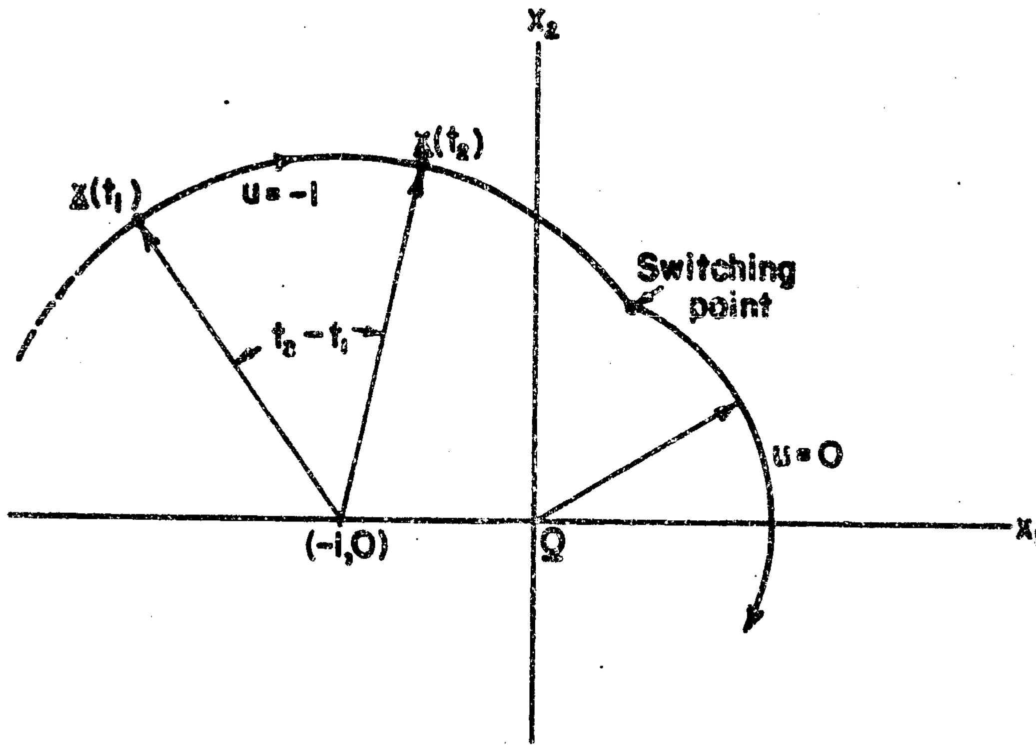


Fig 2 Flügge-Lotz & Marbach



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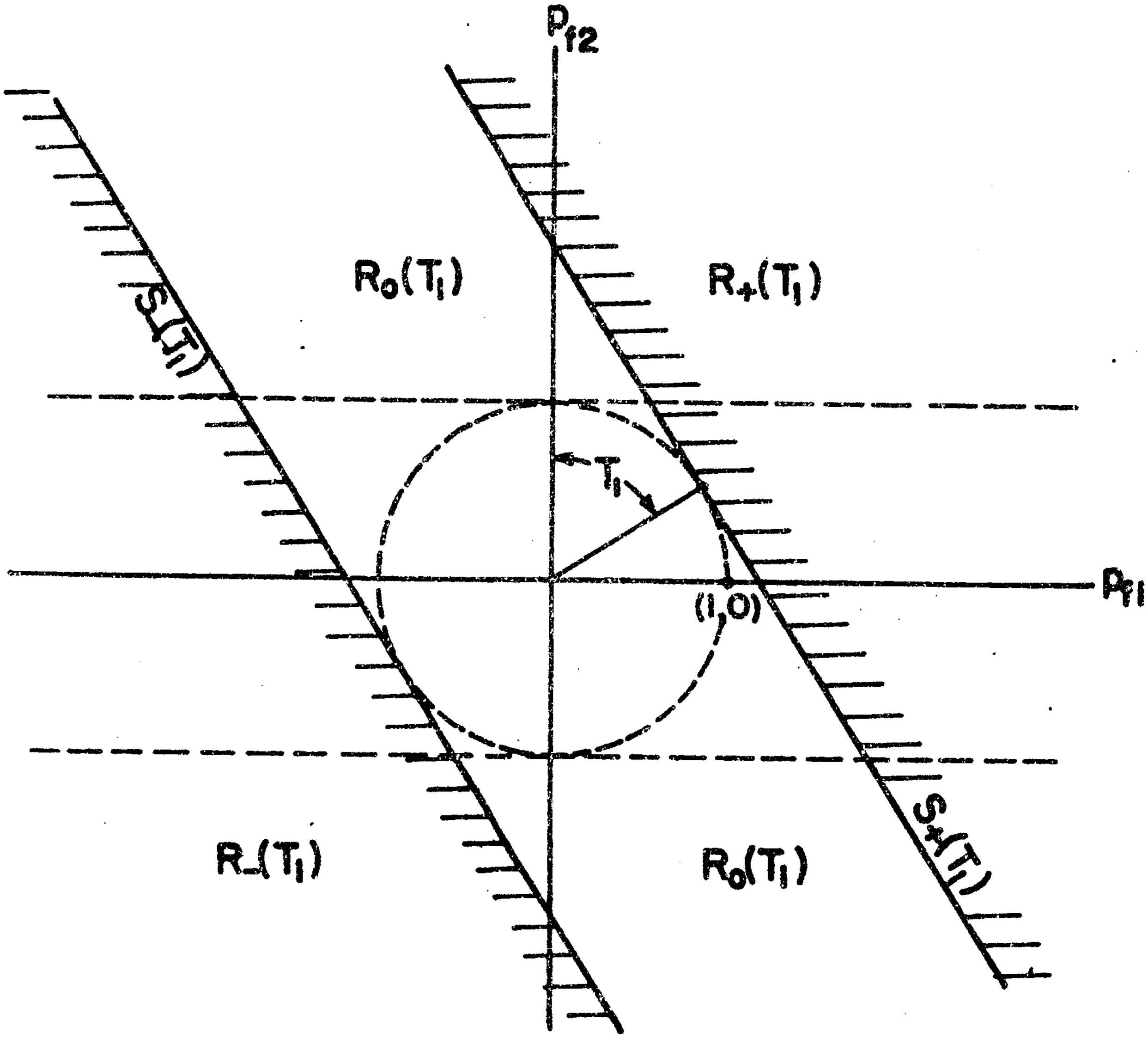


Fig 4 . Flugge-Lotz & MARBACH

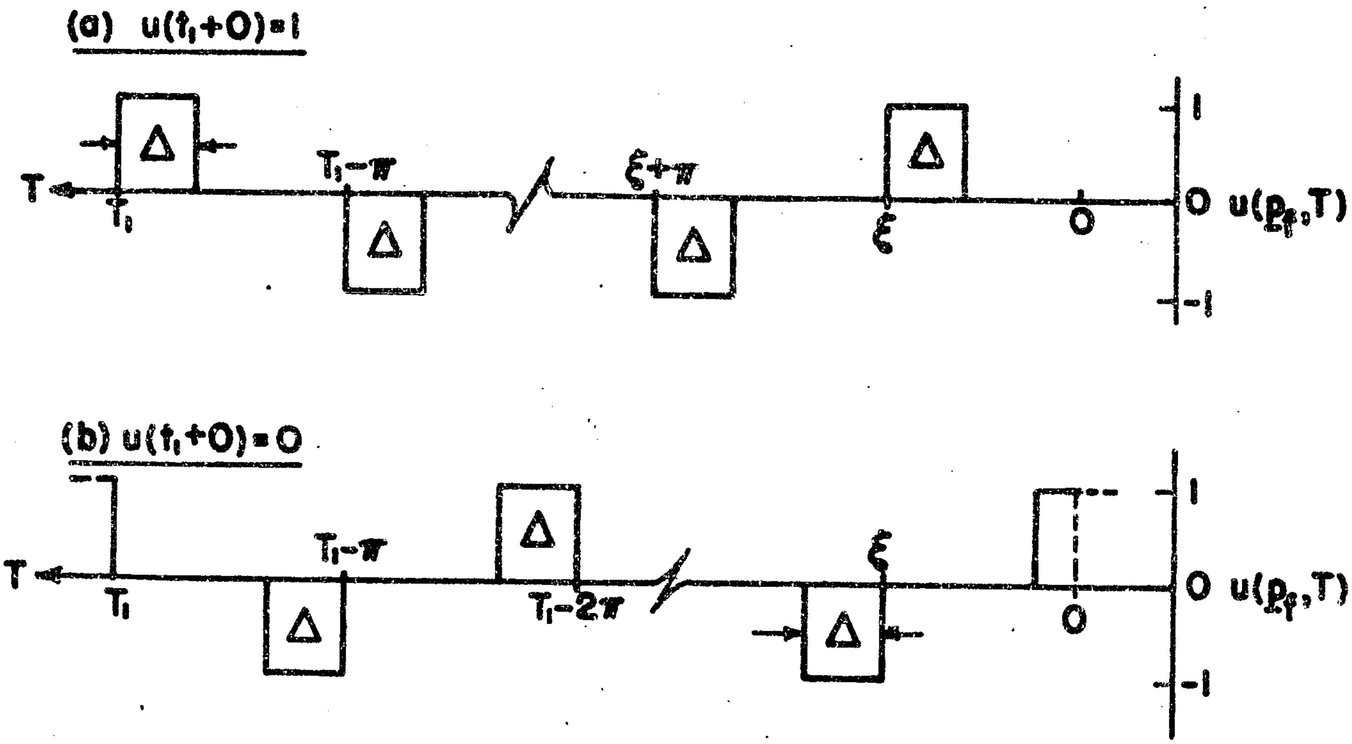


Fig 5 - Flügge-Lotz & Marbach

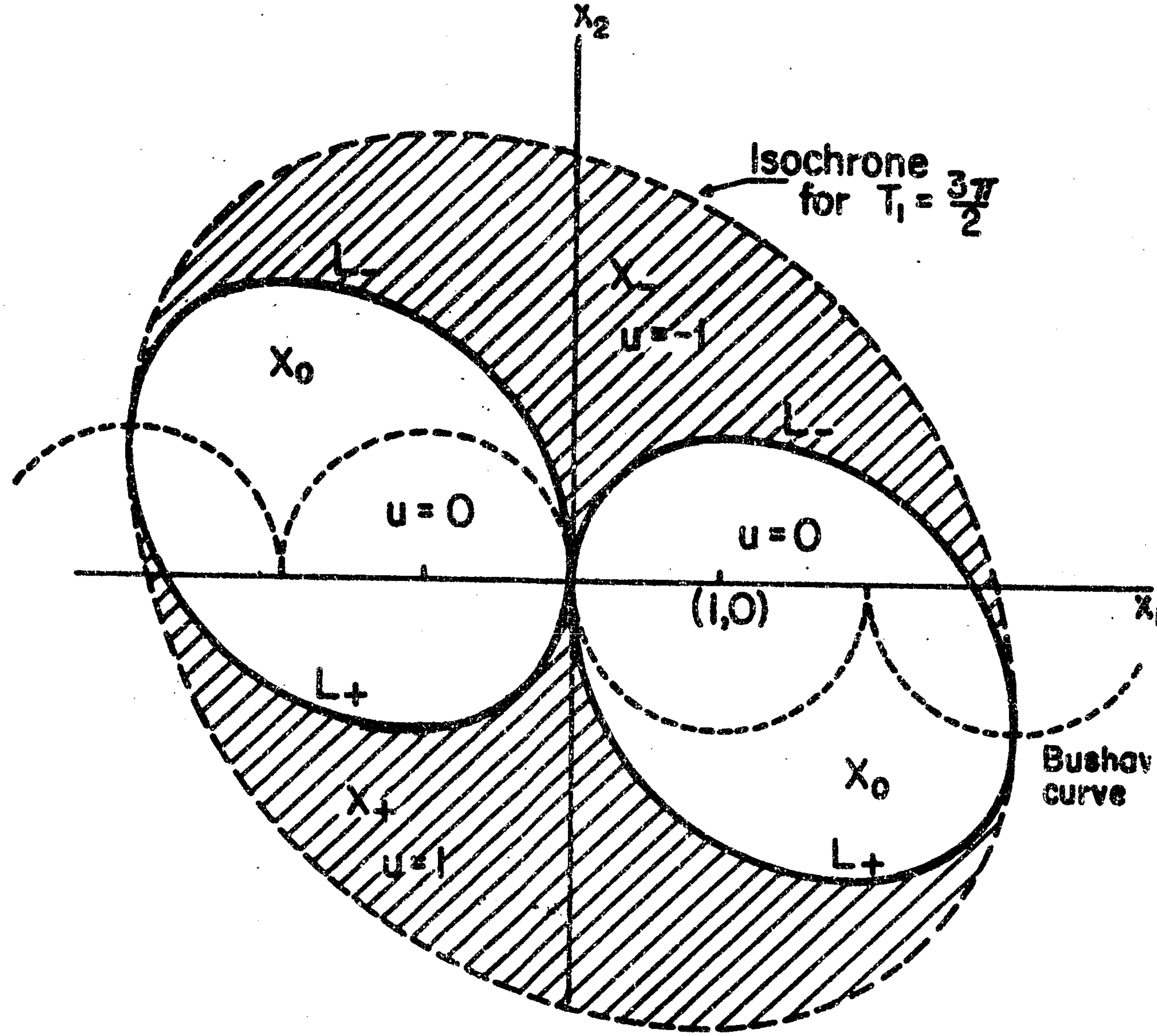


FIG 6 - Flügge-Lotz & MARBACH

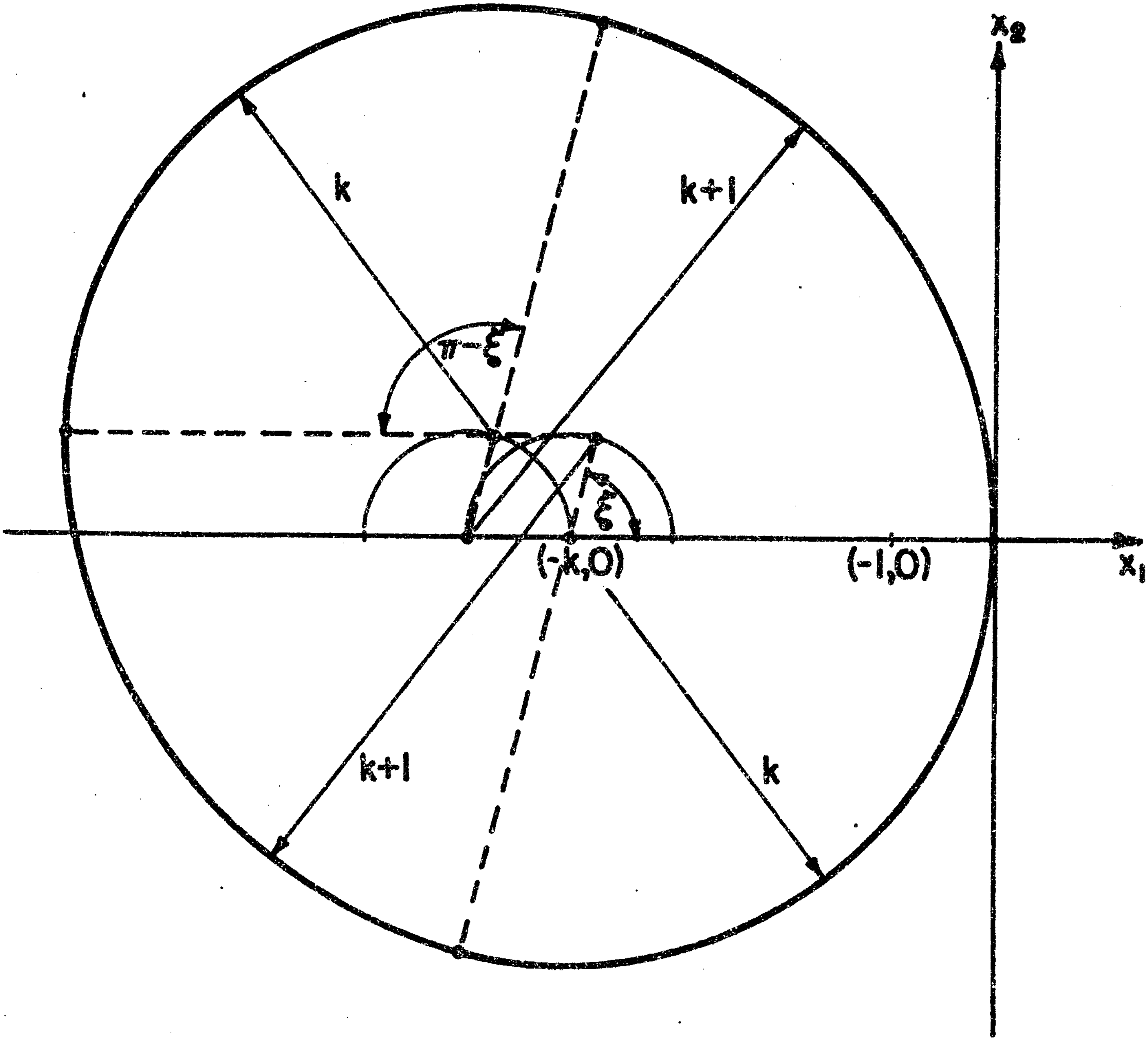


FIG 7 - FLÜGGE-LOTZ & MARBACH

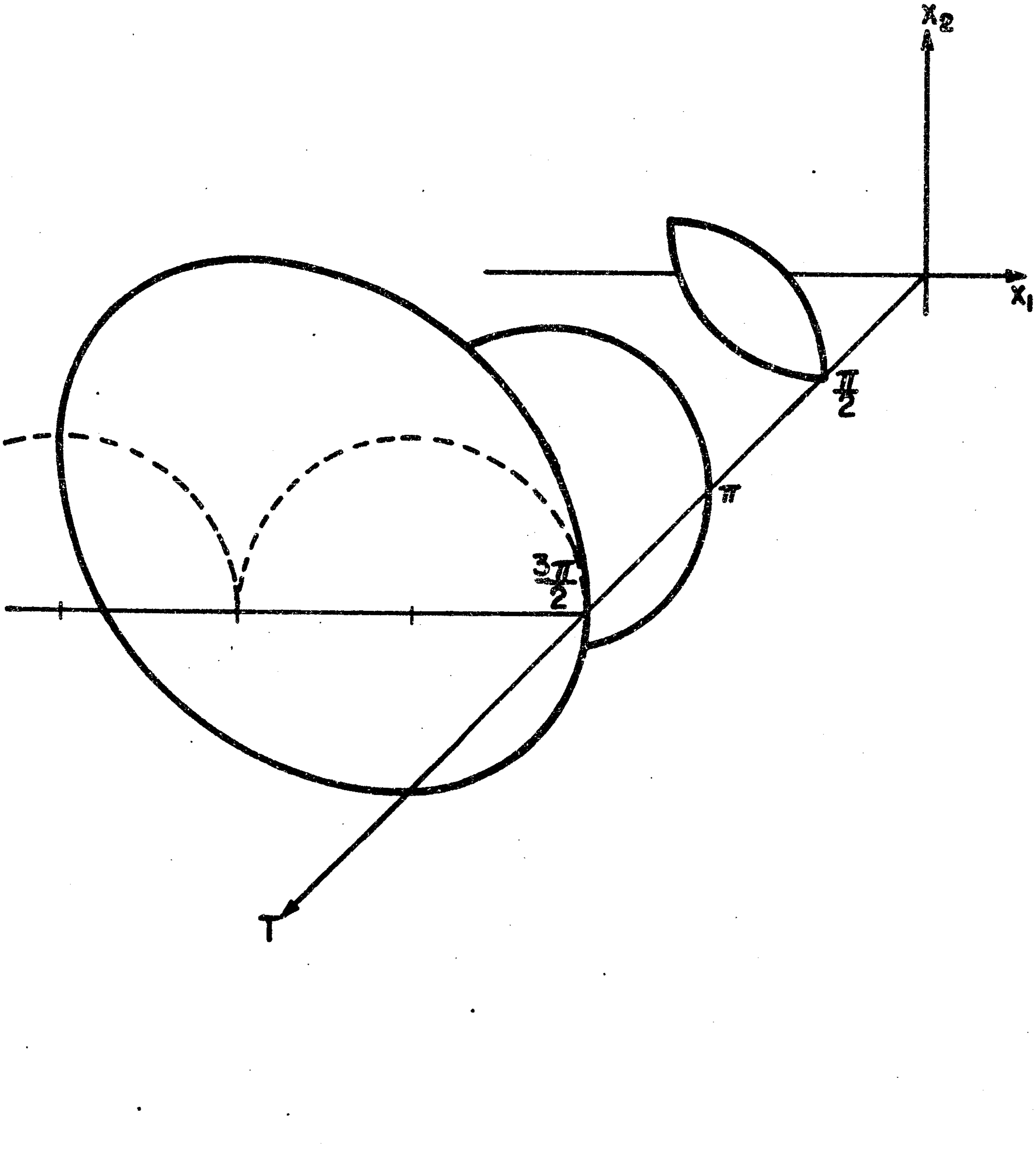


Fig 8 - Flügge-Lotz & MARBACH

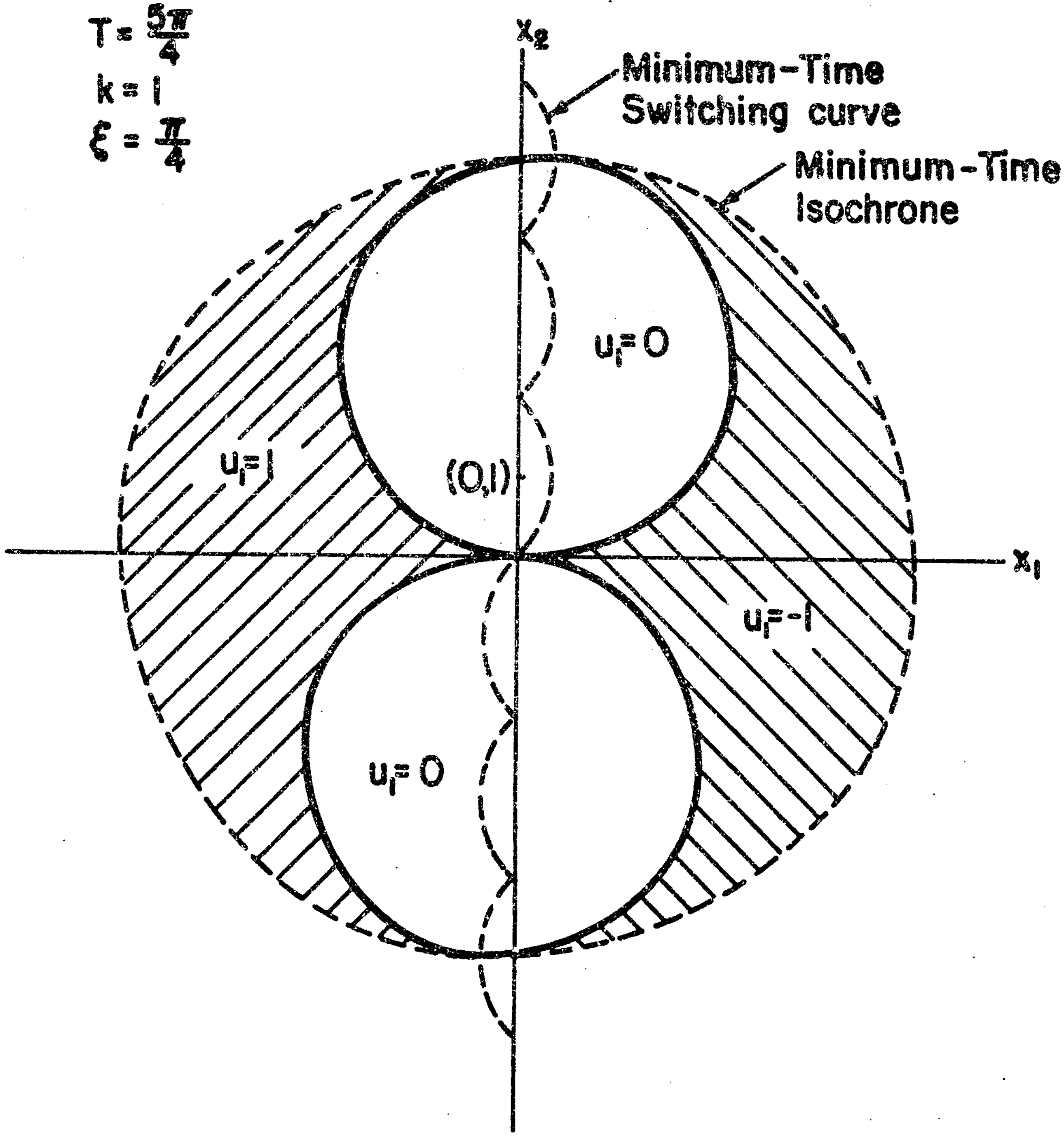
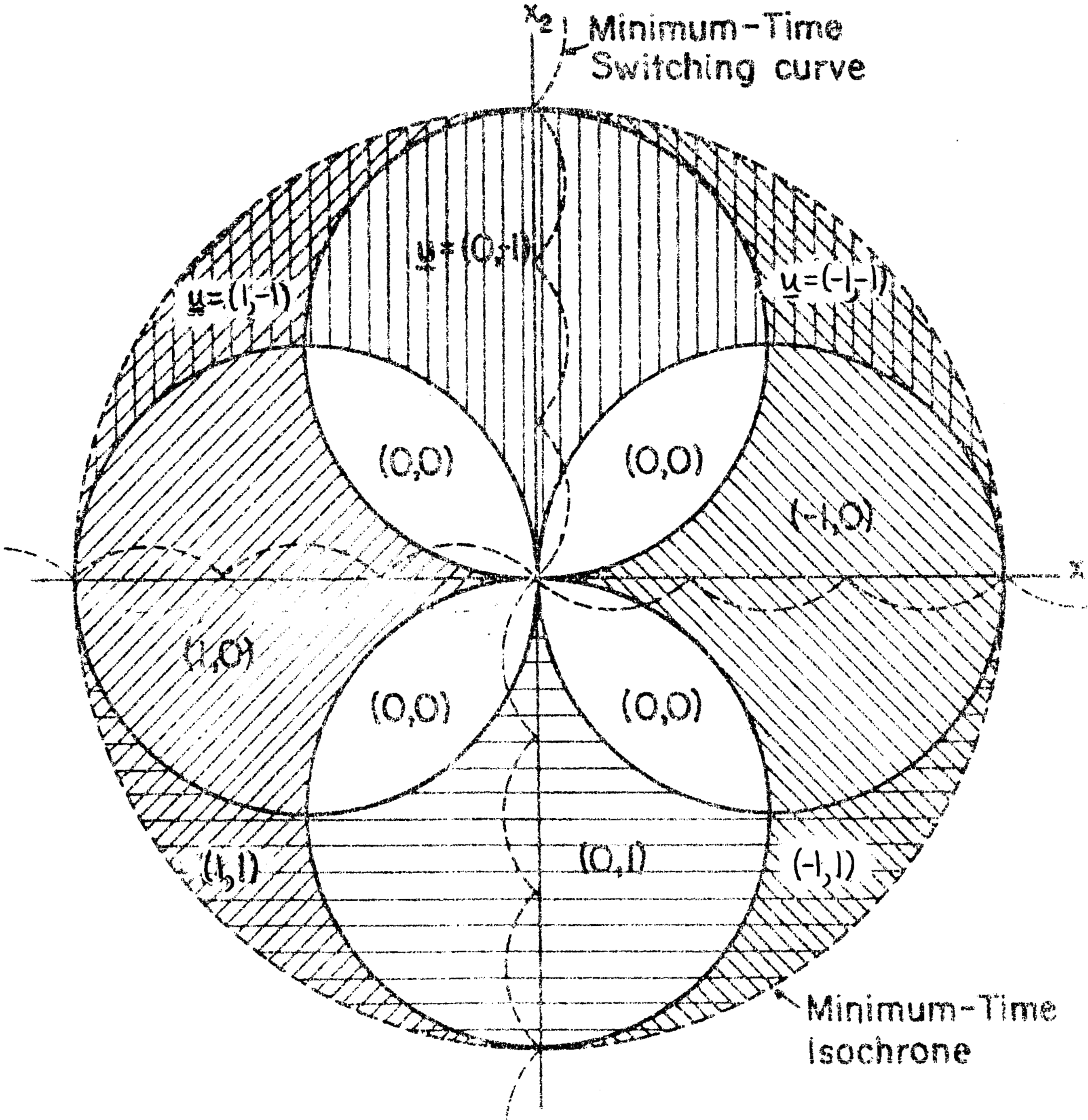


Fig 9 - Flügge-Lotz & Marbach


x_2
Minimum-Time
Switching curve
$\underline{u}=(0,-1)$
$\underline{u}=(1,-1)$
$\underline{u}=(-1,-1)$
(0,0)
(0,0)
(-1,0)
x_1
(1,0)
(0,0)
(0,0)
(1,1)
(0,1)
(-1,1)
Minimum-Time
Isochrone

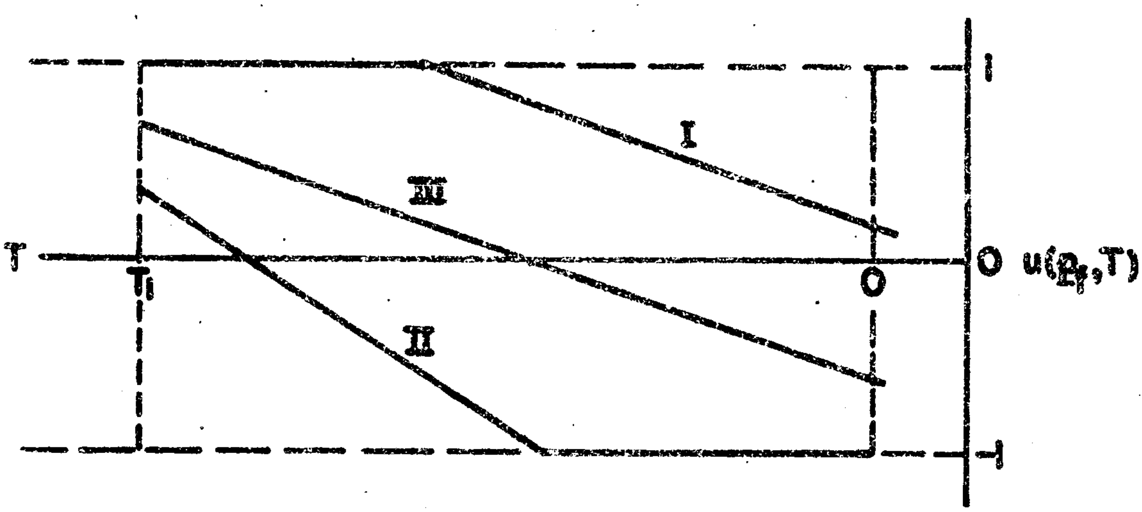


FIG 11 – FLÜGGE-LOTZ & MARBACH

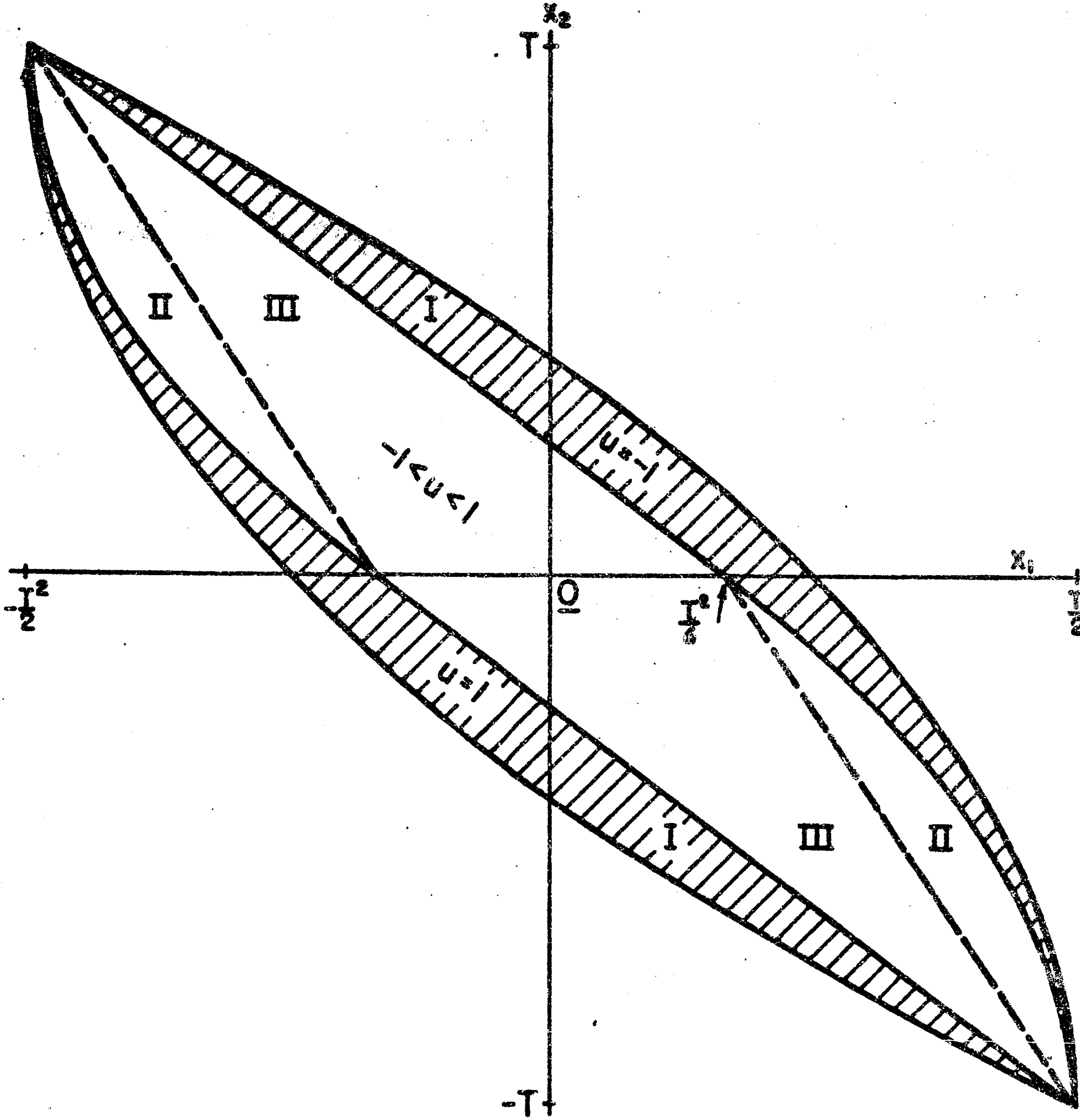


FIG 12 - FLUGGE-LOTZ & MARBACH